國立臺灣師範大學九十六學年度碩士班考試入學招生試題

數學基礎 科試題（資訊工程學系用，本試題共 5 頁）

注意：1.依次序作答，只要標明題號，不必抄題。
2.答案必須寫在答案卷上，否則不予計分。

一、是非題：請用“○”代表“是”，“×”代表“非”。
(10 題，每題 1 分，共 10 分)

For all problems, you may assume all matrices and vectors are of appropriate sizes.

1. If A is *nxn* and det A = 2, then det $A^3 = 6$.
2. If V is a nonzero finite-dimensional vector spaces, and if every set of p elements in V fails to span V, then dim $V > p$.
3. If V is a nonzero finite-dimensional vector spaces, and there exists a linearly dependent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V, then dim $V \leq p$.
4. If dim $V = p$ and Span $S = V$, then S cannot be linearly dependent.
5. Two eigenvectors corresponding to the same eigenvalue are always linearly dependent.
6. If $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
7. An *nxn* matrix with n linearly independent eigenvectors is invertible.
8. If W is a subspace, then $\|proj_w \mathbf{v}\|^2 + \|\mathbf{v} - proj_w \mathbf{v}\|^2 = \|\mathbf{v}\|^2$.
9. If $\hat{\mathbf{x}}$ is a least-squares solution of $A\mathbf{x}=\mathbf{b}$, then $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.
10. If $\{\mathbf{u}, \mathbf{v}\}$ is an orthonormal set in V, then $\|\mathbf{u}-\mathbf{v}\| = \sqrt{2}$.

二、選擇題： 單選題 (5 題，每題 3 分，共 15 分)

1. Define $T: R^2 \to R^2$ by $T(x) = Ax$, where A=$\begin{bmatrix} 23 & -81 \\ 4 & -13 \end{bmatrix}$. Find a basis B for R^2 and the B-matrix D for T with the property that D is an upper triangular matrix.

(A) B=$\left\{\begin{bmatrix} -9 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}, D = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ (B) B=$\left\{\begin{bmatrix} -9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}, D = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$

(C) B=$\left\{\begin{bmatrix} -9 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix}\right\}, D = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix}$ (D) B=$\left\{\begin{bmatrix} -9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}, D = \begin{bmatrix} 5 & 1 \\ 0 & 6 \end{bmatrix}$

2. Consider two bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ for a vector space V such that $b_1 = c_1 - 5c_2$ and $b_2 = 4c_1 - 2c_2$. Find the change-of-coordinates matrix from B to C.

(A) $\begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -5 \\ 4 & -2 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 4 \\ -5 & -2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 4 \\ -5 & -2 \end{bmatrix}$

3. Suppose that the weather in a certain city is either sunny, cloudy, or rainy on a given day, and consider the following:

i. If it is sunny today, there is a 70% chance it will be sunny tomorrow and a 30% chance that it will be cloudy.

ii. If it is cloudy today, there is a 40% chance it will be sunny tomorrow, a 40% chance that it will be cloudy, and a 20% chance that it will be rainy.

iii. If it is rainy today, there is a 40% chance it will be sunny tomorrow, a 30% chance that it will be cloudy, and a 30% chance that it will be rainy.

In the long run, how likely is it that the weather will be rainy on a given day?

(A) 12.1% (B) 11.3% (C) 10.6% (D) 9.5%

4. Let $T: R^2 \to R^2$ be a linear transformation that maps $\mathbf{u}=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ -8 \end{bmatrix}$ and maps $\mathbf{v}=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$. Find the image of $3\mathbf{u}+\mathbf{v}$.

(A) $\begin{bmatrix} 42 \\ -36 \end{bmatrix}$ (B) $\begin{bmatrix} 28 \\ -28 \end{bmatrix}$ (C) $\begin{bmatrix} 14 \\ -12 \end{bmatrix}$ (D) $\begin{bmatrix} 14 \\ 14 \end{bmatrix}$

5. Find a QR factorization of the matrix A, where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

(A) $Q=\begin{bmatrix} 0 & \frac{3}{\sqrt{33}} & \frac{14}{\sqrt{330}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{33}} & \frac{2}{\sqrt{330}} \\ \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{33}} & \frac{9}{\sqrt{330}} \\ \frac{3}{\sqrt{3}} & \frac{-4}{\sqrt{33}} & \frac{7}{\sqrt{330}} \end{bmatrix}, R = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 11 & -3 \\ 0 & 0 & 30 \end{bmatrix}$

(B) $Q=\begin{bmatrix} 0 & 3 & 14 \\ 1 & 2 & 2 \\ -1 & -2 & 9 \\ 1 & -4 & 7 \end{bmatrix}, R = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{11}{\sqrt{33}} & \frac{-3}{\sqrt{33}} \\ 0 & 0 & \frac{30}{\sqrt{330}} \end{bmatrix}$

(C) $Q=\begin{bmatrix} 0 & \frac{3}{\sqrt{33}} & \frac{14}{\sqrt{330}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{33}} & \frac{2}{\sqrt{330}} \\ \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{33}} & \frac{9}{\sqrt{330}} \\ \frac{1}{\sqrt{3}} & \frac{-4}{\sqrt{33}} & \frac{7}{\sqrt{330}} \end{bmatrix}, R = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{11}{\sqrt{33}} & \frac{-3}{\sqrt{33}} \\ 0 & 0 & \frac{30}{\sqrt{330}} \end{bmatrix}$

(D) $Q=\begin{bmatrix} 0 & 3 & 14 \\ 1 & 2 & 2 \\ -1 & -2 & 9 \\ 1 & -4 & 7 \end{bmatrix}, R = \begin{bmatrix} \frac{3}{\sqrt{3}} & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{11}{\sqrt{33}} & 0 \\ 0 & \frac{-3}{\sqrt{33}} & \frac{30}{\sqrt{330}} \end{bmatrix}$

三、簡答題：(9 題，共 75 分)

1. (6 分) If a 6x3 matrix A has rank 3, what is

(a) dim Null A

(b) dim Row A

(c) rank A^T

2. (10 分) Consider the polynomials $p_1(t)=1+t$, $p_2(t)=1-t$, $p_3(t)=4$, $p_4(t)=t+t^2$, $p_5(t)=1+2t+t^2$, and let H be a subspace of $\mathbf{P}^5$, spanned by the set S = $\{p_1, p_2, p_3, p_4, p_5\}$. Find a basis for H in terms of p_i.

3. (9 分) Let P^2 have the inner product given by $<p, q> = \sum_{i=0}^{n} p(t_i)q(t_i)$, where, $p(t)=3t-t^2$, $q(t)=3+2t^2$.

(a) Compute the inner product $<p, q>$ with t at -1, 0, and 1.

(b) Compute $\|p\|$.

(c) Compute the orthogonal projection $\hat{q}$ of q onto the subspace spanned by p.

4. (5 分) A lumberjack has $4n$+110 logs in a pile consisting of n layers. Each layer has two more logs than the layer directly above it. If the top layer has six logs, how many layers are there?

5. (10 分) The famous Ackermann's function can be defined by the recurrence relations

$$A(1,n) = 2^n \qquad \text{for } n \geq 1$$
$$A(m,1) = A(m-1,2) \qquad \text{for } m \geq 2$$
$$A(m,n) = A(m-1, A(m,n-1)) \qquad \text{for } m,n \geq 2$$

Compute (a) $A(3,1)$ and (b) $A(2,3)$.

6. (10 分) Solve the recurrence relation

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n, \quad n \geq 0, a_0 = 0, a_1 = 1, a_2 = 2.$$

7. (10 分) Write the generating functions in closed forms for the sequences (a) $a_n = 3n+4$, and (b) $a_n = n^2$.

8. (5 分) A computer system consists of seven subsystems. Each subsystem might fail independently with a probability of 0.3. The failure of any subsystem will lead to the failure of the whole computer system. Given that the computer system fails, what is the probability that subsystem 1 and only subsystem 1 fails?

9. (10 分)A Graph $G = (V, E)$ is called bipartite if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \phi$, and every edge of G is of the form $\{a,b\}$ with $a \in V_1$ and $b \in V_2$. If each vertex in V_1 is joined with every vertex in V_2, we have a complete bipartite graph. In this case, if $|V_1| = m$, $|V_2| = n$, the graph is denoted by $K_{m,n}$. Let $m,n \in Z^+$ with $m \geq n \geq 2$. (a) Determine how many distinct cycles of length 4 there are in $K_{m,n}$. (b) How many different paths of length 2 are there in $K_{m,n}$?